Wave Propagation in Overhead Wires with Ground Return

By JOHN R. CARSON

I

THE problem of wave propagation along a transmission system composed of an overhead wire parallel to the (plane) surface of the earth, in spite of its great technical importance, does not appear to have been satisfactorily solved.[1] While a complete solution of the actual problem is impossible, on account of the inequalities in the earth's surface and its lack of conductive homogeneity, the solution of the problem, where the actual earth is replaced by a plane homogeneous semi-infinite solid, is of considerable theoretical and practical interest. The solution of this problem is given in the present paper, together with formulas for calculating inductive disturbances in neighboring transmission systems.

The axis of the wire is taken parallel to the z-axis at height h above the xz-plane and passes through the y-axis at point O' as shown in Fig. 1 herewith. The "image" of the wire is designated by O''.

For $y>0$ (in the dielectric) the medium is supposed to have zero conductivity, while for $y<0$ (in the ground) the conductivity of the medium is designated by λ. The xz-plane represents the surface of separation between dielectric and ground.

We consider a wave propagated along the z-axis and the current, charge and field are supposed to contain the common factor $\exp(-\Gamma z+i\omega t)$, which, however, will be omitted for convenience in the formulas. The propagation constant, Γ, is to be determined. It is assumed, *ab initio*, as a very small quantity in c.g.s. units.[2]

In the ground ($y\leqq 0$) the axial electric force is formulated as the

[1] See Rudenberg, Zt. f. Angewandt, Math. u. Mechanik, Band 5, 1925. In that paper the current density in the ground is assumed to be distributed with radial symmetry. The resulting formulas are not in agreement with those of the present paper. Since this paper was set up in type I have learned that formulas equivalent to equations (26), (28), (31) for the *mutual impedance* of two parellel wires were obtained by my colleague, Dr. G. A. Campbell, in 1917. It is to be hoped that his solution will be published shortly.

[2] The simplifying assumptions introduced in this analysis are essentially the same as those employed and discussed in "Wave Propagation Over Parallel Wires: The Proximity Effect," *Phil. Mag.*, Vol. xli, April, 1921.

general solution, symmetrical with respect to x, of the wave equation; thus

$$E_z = -\int_0^\infty F(\mu) \cos x\mu \, e^{y\sqrt{\mu^2+i\alpha}} d\mu, \qquad y \leq 0 \tag{1}$$

where

$\alpha = 4\pi\lambda\omega$,
λ = conductivity of ground in elm. c.g.s. units,
$i = \sqrt{-1}$,
$\omega/2\pi$ = frequency in cycles per second.

(In the following analysis and formulas, elm. c.g.s. units are employed throughout).

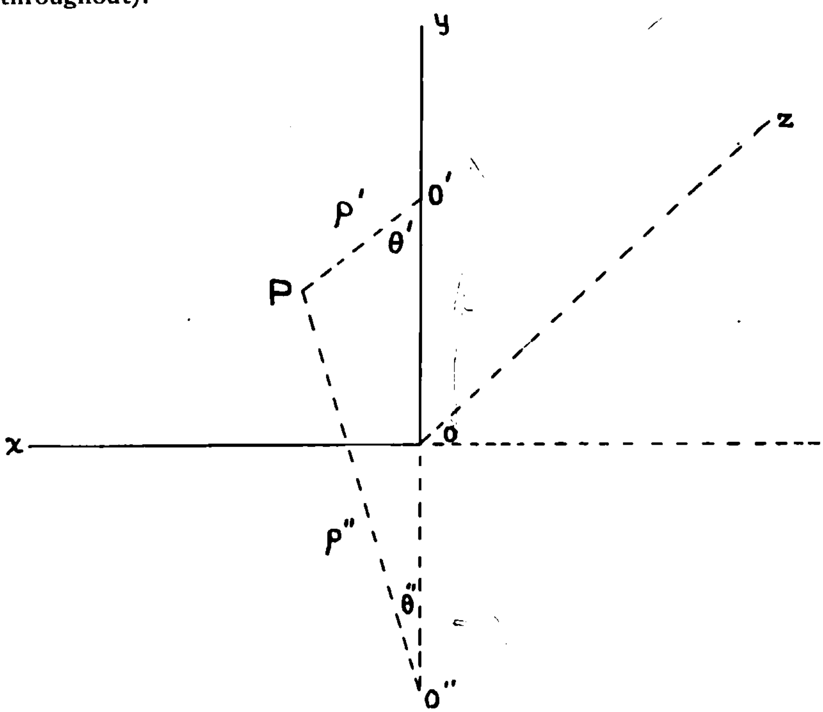

Fig. 1

Assuming that in the ground E_x and E_y are negligible compared with E_z, we have from the formula, curl $E = -\frac{\partial}{\partial t} H$,

$$i\omega H_x = -\frac{\partial}{\partial y} E_z$$

$$i\omega H_y = \frac{\partial}{\partial x} E_z.$$

Whence, in the ground

$$H_x=\frac{1}{i\omega}\int_0^\infty \sqrt{\mu^2+i\alpha}\,.F(\mu).\cos x\mu.e^{y\sqrt{\mu^2+i\alpha}}d\mu, \tag{2}$$

$$H_y=\frac{1}{i\omega}\int_0^\infty \mu.F(\mu).\sin x\mu.e^{y\sqrt{\mu^2+i\alpha}}d\mu, \tag{3}$$

it being understood that $y\leqq 0$. The function $F(\mu)$ in the preceding formulas is to be determined by the boundary conditions.

In the dielectric, H_x and H_y may be regarded as composed of two components; thus

$$H_x=H_x^0+H_x',$$

$$H_y=H_y^0+H_y',$$

where H_x^0, H_y^0 designate the field due to the current I in the wire, and H_x', H_y' the field of the ground current.

Neglecting axial displacement currents in the dielectric, and assuming that the wire is of sufficiently small radius so that the distribution of current over its cross section is symmetrical, we have

$$H_x^0=\frac{\cos\Theta'}{\rho'}.\ 2I, \tag{4}$$

$$H_y^0=\frac{\sin\Theta'}{\rho'}.\ 2I, \tag{5}$$

where

$$\rho'=\sqrt{x^2+(y-h)^2},$$

$$\cos\Theta'=\frac{h-y}{\rho'}, \tag{6}$$

$$\sin\Theta'=x/\rho'.$$

The secondary magnetic field H_x', H_y' is taken as

$$H_x'=\int_0^\infty \phi(\mu)\cos x\mu.e^{-y\mu}d\mu, \tag{7}$$

$$H_y'=-\int_0^\infty \phi(\mu)\sin x\mu.e^{-y\mu}d\mu. \tag{8}$$

At the surface of separation $y=0$, H_x^0, H_y^0 are expressible as the Fourier integrals

$$H_x^0=2I\int_0^\infty \cos x\mu.e^{-h\mu}d\mu, \tag{9}$$

$$H_y^0=2I\int_0^\infty \sin x\mu.e^{-h\mu}d\mu. \tag{10}$$

Also at the surface of separation of the two media ($y=0$), H_x and H_y must be continuous. Equating the values of H_x and H_y at $y=0$, as given by (2), (3) and by (7), (8) and (9), (10), we have

$$\frac{1}{i\omega}\sqrt{\mu^2+i\alpha}.F(\mu)=2I.e^{-h\mu}+\phi(\mu),$$

$$\frac{1}{i\omega}\mu.F(\mu)=2I.e^{-h\mu}-\phi(\mu),$$

whence

$$F(\mu)=\frac{i\omega e^{-h\mu}}{\sqrt{\mu^2+i\alpha}+\mu}4I, \tag{11}$$

$$\phi(\mu)=\frac{(\sqrt{\mu^2+i\alpha}-\mu)}{\sqrt{\mu^2+i\alpha}+\mu}e^{-h\mu}.2I, \tag{12}$$

which determines the functions $F(\mu)$ and $\phi(\mu)$.

Inserting the value of $F(\mu)$, as given by (11) in (1), the axial electric force E_z in the ground ($y\leqq 0$) and therefore the distribution of current density in the ground is expressed as a Fourier integral in terms of the frequency $\omega/2\pi$, the current I in the wire, the height h of the wire above ground, and the conductivity λ of the ground. Similarly the insertion of $\phi(\mu)$, as given by (12) in formulas (7) and (8) gives the magnetic field H_x, H_y in the dielectric. Thus

$$E_z(x,y)=E_z=-i4\omega I\int_0^\infty \frac{e^{-\mu h}}{\sqrt{\mu^2+i\alpha}+\mu}e^{y\sqrt{\mu^2+i\alpha}}\cos x\mu.d\mu, \qquad y\leqq 0. \tag{13}$$

This can be further simplified if we write

$$x'=x\sqrt{\alpha}$$

$$y'=y\sqrt{\alpha}$$

$$h'=h\sqrt{\alpha},$$

whence

$$E_z=-4\omega I\int_0^\infty (\sqrt{\mu^2+i}-\mu)e^{-h'\mu}.e^{y'\sqrt{\mu^2+i}}\cos x'\mu\, d\mu, \qquad y\leqq 0. \tag{14}$$

The axial electric force in the dielectric is now to be formulated. This is always derivable from a vector and a scalar potential; thus

$$E_z=-i\omega A_z-\frac{\partial}{\partial z}V, \tag{15}$$

where A_z is the vector potential of the axial currents, and V the scalar potential. Consequently,

$$E_z(x,y)-E_z(x,0)=-i\omega(A_z(x,y)-A_z(x,0))-\frac{\partial}{\partial z}(V(x,y)-V_0). \tag{16}$$

Here $E_z(x,0)$ is the axial electric intensity at the surface of the ground plane ($y=0$), and

$$A_z(x,y)-A_z(x,0)=\int_0^y H_x(x,y)dy. \tag{17}$$

$V(x,y)-V_0$ is the difference in the scalar potential between the point x,y and the ground, which is due to the charges on the wire and on the surface of the ground. For convenience, it will be designated by V.

By means of (16) and the preceding formulas we get [3]

$$\begin{aligned} E_z=&-4\omega I\int_0^\infty(\sqrt{\mu^2+i}-\mu)e^{-(h'+y')\mu}\cos x'\mu\, d\mu \\ &-i\,2\omega I\log(\rho''/\rho')-\frac{\partial}{\partial z}V, \qquad y\geqq 0 \end{aligned} \tag{18}$$

where

$$\rho'=\sqrt{(h-y)^2+x^2}$$
$=$distance of point x,y from wire,
$$\rho''=\sqrt{(h+y)^2+x^2}$$
$=$distance of point x,y from image of wire.

The first two terms on the right hand side of (18) represent the electric force due to the varying magnetic field; the term $-\frac{\partial}{\partial z}V$ represents the axial electric intensity due to the charges on the surface of the wire and the ground. If Q be the charge per unit length, V is calculable by usual electrostatic methods on the assumption that the surface of the wire and the surface of the ground are equipotential surfaces, and their difference of potential is Q/C where C is the electrostatic capacity between wire and ground.[4]

II

By aid of the preceding analysis and formulas, we are now in a position to derive the propagation constant, Γ, and characteristic impedance, K, which characterize wave propagation along the system. Let z denote the "internal" or "intrinsic" impedance of the wire per

[3] As a check on this formula note that together with (14) it satisfies the condition of continuity of E_z at $y=0$.

[4] See "Wave Propagation Over Parallel Wires: The Proximity Effect," *Phil. Mag.*, Vol. xli, Apr., 1921.

unit length. (With small error this may usually be taken as the resistance per unit length of the wire.) The axial electric intensity at the surface of the wire is then zI. Equating this to the axial electric intensity at the surface of the wire as given by (18) and replacing $\partial/\partial z$ by $-\Gamma$, we have

$$zI = -4\omega I \int_0^\infty (\sqrt{\mu^2+i}-\mu)e^{-2h'\mu}d\mu \\ -i\,2\omega I.\log(\rho''/a)+\Gamma V. \tag{19}$$

Writing $V=Q/C$ and

$$i\omega Q = \Gamma I - GV = \Gamma I - \frac{G}{C}Q,$$

where G is the leakage conductance to ground per unit length, we have, solving for Γ,

$$\Gamma^2 = (G+i\omega C)[z+i2\omega.\log(\rho''/a)]+4\omega\int_0^\infty (\sqrt{\mu^2+i}-\mu)e^{-2h'\mu}d\mu). \tag{20}$$

Writing this in the usual form

$$\Gamma^2 = (R+iX)(G+i\omega C), \tag{21}$$

the characteristic impedance is given by

$$K^2 = \frac{R+iX}{G+i\omega C} \tag{22}$$

and the *series impedance per unit length of the circuit* is

$$R+iX = Z = z+i2\omega.\log(\rho''/a)+4\omega\int_0^\infty (\sqrt{\mu^2+i}-\mu)e^{-2h'\mu}d\mu. \tag{23}$$

It will be observed that the first two terms on the right hand side of (23) represent the series impedance of the circuit *if the ground is a perfect conductor*; the infinite integral formulates the effect of the finite conductivity of the ground.

The *mutual impedance* [5] Z_{12} between two parallel ground return circuits with wires at heights h_1 and h_2 above ground and a separation x between their vertical planes is given by

$$Z_{12} = i2\omega.\log(\rho''/\rho')+4\omega\int_0^\infty (\sqrt{\mu^2+i}-\mu)e^{-(h_1'+h_2')\mu}\cos x'\mu\, d\mu, \tag{24}$$

[5] It will be noted that the mutual impedance is equal to the axial electric intensity at the axis of the second wire due to the varying magnetic field of unit current in the first wire and its accompanying distribution of ground current.

where

$$\rho''=\sqrt{(h_1+h_2)^2+x^2}$$
$$\rho'=\sqrt{(h_1-h_2)^2+x^2}$$
$$h_1'=h_1\sqrt{\alpha}$$
$$h_2'=h_2\sqrt{\alpha}$$
$$x'=x\sqrt{\alpha}.$$

From the preceding the series *self impedance* of the ground return circuit may be conveniently written as

$$Z=Z^0+Z' \tag{25}$$

and the *mutual impedance* as

$$Z_{12}=Z_{12}^0+Z_{12}' \tag{26}$$

where Z^0, Z_{12}^0 are the self and mutual impedances respectively, ***on the assumption of a perfectly conducting ground***, and

$$Z'=4\omega\int_0^\infty(\sqrt{\mu^2+i}-\mu)e^{-2h'\mu}d\mu, \tag{27}$$

$$Z_{12}'=4\omega\int_0^\infty(\sqrt{\mu^2+i}-\mu)e^{-(h_1'+h_2')\mu}\cos x'\mu\, d\mu. \tag{28}$$

The calculation of the circuit constants and the electromagnetic field in the dielectric depends, therefore, on the evaluation of an infinite integral of the form

$$J(p,q)=J=\int_0^\infty(\sqrt{\mu^2+i}-\mu)e^{-p\mu}.\cos q\mu\, d\mu. \tag{29}$$

In terms of this integral

$$Z'=4\omega.J(2h'.0) \tag{30}$$

$$Z_{12}'=4\omega.J(h_1'+h_2',\, x'). \tag{31}$$

To the solution of the infinite integral $J(p,q)$ we now proceed.

III

The solution of equation (29), that is, the evaluation of $J(p,q)$ can be made to depend on the solution of the infinite integral

$$\int_0^\infty\sqrt{\mu^2+\alpha^2}.e^{-\beta\mu}d\mu$$

which has been worked out and communicated to me by R. M. Foster. It is

$$\frac{\alpha}{\beta}\left\{K_1(\alpha\beta)+G(\alpha\beta)\right\}$$

where $K_1(x)$ is the Bessel function of the second kind and first order as defined by Jahnke und Emde, Funktionentafeln, pg. 93, and $G(x)$ is the absolutely convergent series

$$G(x)=\frac{x^2}{3}-\frac{x^4}{3^2.5}+\frac{x^6}{3^2.5^2.7}-\ \ldots\ .$$

On the basis of this solution, it is a straightforward though intricate and tedious process to derive the following solution for $J(p,q)$ of equation (29).

Writing $r=\sqrt{p^2+q^2}$ and $\Theta=\tan^{-1}(q/p)$, it is $J=P+iQ$

in which

$$P=\frac{\pi}{8}(1-s_4)+\frac{1}{2}\left(\log\frac{2}{\gamma r}\right)s_2+\frac{1}{2}\Theta.s_2' \qquad (32)$$
$$-\frac{1}{\sqrt{2}}\sigma_1+\frac{1}{2}\sigma_2+\frac{1}{\sqrt{2}}\sigma_3,$$

$$Q=\frac{1}{4}+\frac{1}{2}\left(\log\frac{2}{\gamma r}\right)(1-s_4)-\frac{1}{2}\Theta.s_4' \qquad (33)$$
$$+\frac{1}{\sqrt{2}}\sigma_1-\frac{\pi}{8}s_2+\frac{1}{\sqrt{2}}\sigma_3-\frac{1}{2}\sigma_4.$$

In these equations log γ is Euler's constant:

$\gamma=1.7811$, $\log\dfrac{2}{\gamma}=0.11593$, $\log\gamma=0.57722$ and σ_1, σ_2, σ_3, σ_4, s_2, s_2', s_4, s_4', are infinite series defined as follows:

$$s_2=\frac{1}{1!2!}\left(\frac{r}{2}\right)^2\cos 2\Theta-\frac{1}{3!4!}\left(\frac{r}{2}\right)^6\cos 6\Theta+\ \ldots\ ,$$

$$s_2'=\frac{1}{1!2!}\left(\frac{r}{2}\right)^2\sin 2\Theta-\frac{1}{3!4!}\left(\frac{r}{2}\right)^6\sin 6\Theta+\ \ldots\ ,$$

$$s_4=\frac{1}{2!3!}\left(\frac{r}{2}\right)^4\cos 4\Theta-\frac{1}{4!5!}\left(\frac{r}{2}\right)^8\cos 8\Theta+\ \ldots\ .$$

$$s_4' = \frac{1}{2!3!}\left(\frac{r}{2}\right)^4 \sin 4\Theta - \frac{1}{4!5!}\left(\frac{r}{2}\right)^8 \sin 8\Theta + \ldots,$$

$$\sigma_1 = \frac{r\cos\Theta}{3} - \frac{r^5 \cos 5\Theta}{3^2.5^2.7} + \frac{r^9 \cos 9\Theta}{3^2.5^2.7^2.9^2.11} - \ldots,$$

$$\sigma_3 = \frac{r^3\cos 3\Theta}{3^2.5} - \frac{r^7 \cos 7\Theta}{3^2.5^2.7^2.9} + \frac{r^{11} \cos 11\Theta}{3^2.5^2.7^2.9^2.11^2.13} - \ldots,$$

$$\sigma_2 = \left(1+\frac{1}{2}-\frac{1}{4}\right)\frac{1}{1!2!}\left(\frac{r}{2}\right)^2 \cos 2\Theta$$

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{8}\right)\frac{1}{3!4!}\left(\frac{r}{2}\right)^6 \cos 6\Theta + \ldots,$$

$$= \frac{5}{4} s_2 \text{ approximately},$$

$$\sigma_4 = \left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{6}\right)\frac{1}{2!3!}\left(\frac{r}{2}\right)^4 \cos 4\Theta$$

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{10}\right)\frac{1}{4!5!}\left(\frac{r}{2}\right)^8 \cos 8\Theta + \ldots.$$

$$= \frac{5}{3} s_4 \text{ approximately}.$$

It is to be regretted that the foregoing formulas appear so complicated. The series, however, are very rapidly convergent and for $r \leqq 2$ only the two leading terms of each series need be retained. For $r \leqq 1$, only the leading terms are of importance.

For the important range $r \leqq 1/4$,

$$P = \frac{\pi}{8} - \frac{1}{3\sqrt{2}} r\cos\Theta + \frac{r^2}{16}\cos 2\Theta\left(0.6728 + \log\frac{2}{r}\right) + \frac{r^2}{16}\Theta\sin 2\Theta, \quad (34)$$

$$Q = -0.0386 + \frac{1}{2}\log\left(\frac{2}{r}\right) + \frac{1}{3\sqrt{2}} r\cos\Theta. \quad (35)$$

For $r > 5$ the following asymptotic expansions, derivable from (29) by repeated partial integrations, are to be employed.

$$P = \frac{1}{\sqrt{2}}\frac{\cos\Theta}{r} - \frac{\cos 2\Theta}{r^2} + \frac{1}{\sqrt{2}}\frac{\cos 3\Theta}{r^3} + \frac{3}{\sqrt{2}}\frac{\cos 5\Theta}{r^5} \ldots, \quad (36)$$

$$r > 5.$$

$$Q = \frac{1}{\sqrt{2}}\frac{\cos\Theta}{r} - \frac{1}{\sqrt{2}}\frac{\cos 3\Theta}{r^3} + \frac{3}{\sqrt{2}}\frac{\cos 5\Theta}{r^5} - \ldots, \quad (37)$$

For large values of $r(r>10)$, these reduce to

$$J=\frac{1+i}{\sqrt{2}}\frac{\cos\Theta}{r}-\frac{\cos 2\Theta}{r^2},\ r>10. \tag{38}$$

In view of the somewhat complicated character of the function in the range $1/4 \leqq r \leqq 5$ the curves shown below have been computed. These show $J=P+iQ$ as a function of r for $\Theta=0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}$. By interpolation it is possible to estimate with fair accuracy the value of the functions for intermediate values of Θ.

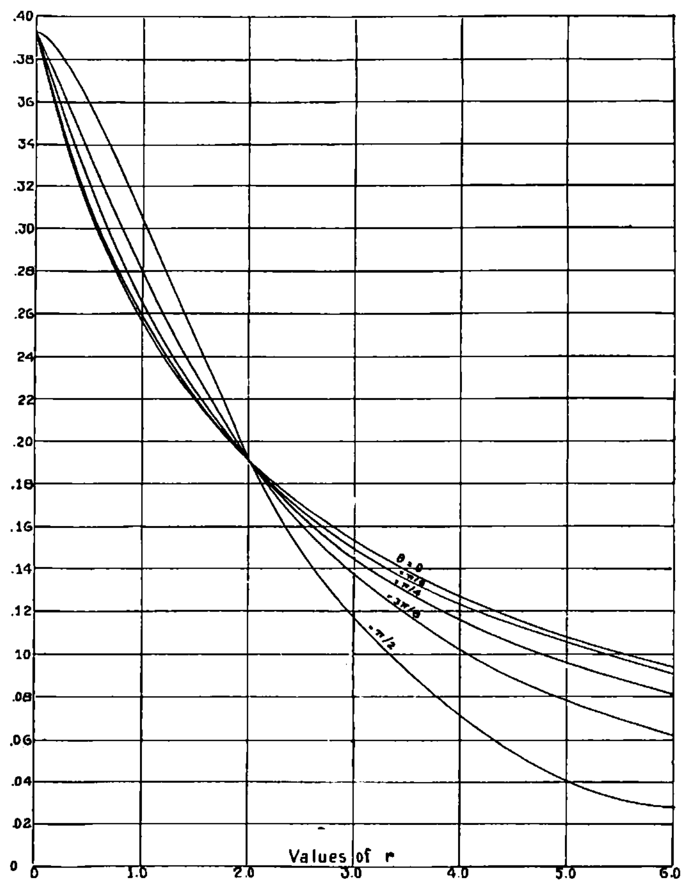

Fig. 2 P=real part of J

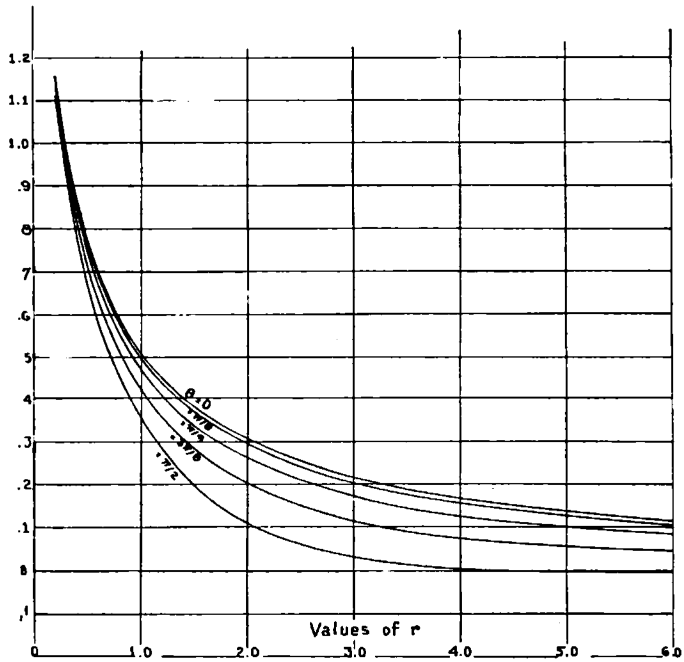

Fig. 3 Q = imaginary part of J

IV

The theory and formulas of the preceding sections will now be reviewed and summarized as regards their principal applications to technical transmission problems where the ground forms, in whole or part, the "return" part of the circuit. In such problems we are interested in the electric intensity in the dielectric and in the ground, and in particular in the self impedance and mutual impedances of ground return circuits. The calculation of these quantities is provided for by the general analysis and the solution of the infinite integral J. Reference should be made to Fig. 1 shown in section I for the geometry of the system and coordinate system employed.

1. *The Axial Electric Intensity E_z in the Dielectric.* (See equations (15) and (18)).

$$E_z = -\frac{\partial}{\partial z} V - (i2\omega \log (\rho''/\rho') + 4\omega J) I$$

where

$$\rho' = \sqrt{(h-y)^2+x^2}$$

$$= \text{distance of point in dielectric } \textit{from axis} \text{ of wire.}$$

$$\rho'' = \sqrt{(h+y)^2+x^2}$$

$$= \text{distance of point in dielectric } \textit{from image} \text{ of wire.}$$

$$r = \rho''\sqrt{\alpha}$$

$$\Theta = \sin^{-1}(x/\rho'')$$

$$\alpha = 4\pi\lambda\omega.$$

These values of r and Θ are, of course, to be employed in calculating $J=P+iQ$ from the formulas and curves of the preceding section. As a special case *the electric intensity at the surface of the earth is*

$$E_s = -4\omega JI$$

$$\rho'' = \sqrt{h^2+x^2}$$

$$r = \rho''\sqrt{\alpha}$$

$$\Theta = \sin^{-1}(x/\rho'')$$

2. *Self Impedance of Ground Return Circuit.* (See equations (25), (27), (30)).

$$Z = Z^0 + 4\omega J$$

$$Z^0 = \text{self impedance with perfectly conducting ground.}$$

$$r = 2h\sqrt{\alpha}$$

$$\Theta = 0.$$

3. *Mutual Impedance of Ground Return Circuits.* (See equations (26), (28), (31)).

$$Z_{12} = Z_{12}^0 + 4\omega J$$

$$Z_{12}^0 = \text{mutual impedance with perfectly conducting ground.}$$

$$r = \sqrt{\alpha}\sqrt{(h_1+h_2)^2+x^2} = \rho''\sqrt{\alpha}$$

$$\Theta = \sin^{-1}(x/\rho'').$$

The axial electric intensity E_z in the ground ($y<0$) is given by equation (1), and the subsequent analysis, whence

$$E_z = -4\omega I\int_0^\infty (\sqrt{\mu^2+i}-\mu).\cos x'\mu . e^{-(h'\mu+g'\sqrt{\mu^2+i})}d\mu$$

where, as before

$$x' = x\sqrt{\alpha}$$

$$h' = h\sqrt{\alpha}$$

and

$$g' = \sqrt{\alpha} \text{ times the depth below the surface of the ground.}$$

$$= g\sqrt{\alpha}.$$

The integral can undoubtedly be evaluated in somewhat the same way as (29) and can in any case be numerically computed without much difficulty. Owing, however, to the secondary technical interest in the electric intensity below the surface of the earth, the detailed solution has not been undertaken, nor has the magnetic field been worked out.

V

The practical utility of the preceding theory and formulas will now be illustrated by a brief sketch of their application to two important transmission problems.

The Wave Antenna

When a transmission line with "ground return" is employed as a radio receiving antenna it is called a wave antenna. The theory and design of such an antenna requires a knowledge of the transmission characteristics of the ground return circuit, which are calculable, as shown above, from the geometry and constants of the overhead wire, together with $Z' = 4\omega J$, which may be termed the "ground return" impedance.

We assume that the wire is approximately 30 ft. above the ground ($h = 10^3$) and that the frequency is 5.10^4 c.p.s. corresponding to the frequency employed in Trans-Atlantic radio communication. The ground conductivity λ is exceedingly variable, depending on the locality and weather conditions. Calculations of Z' will therefore be made for two extreme cases, $\lambda = 10^{-12}$ and $\lambda = 10^{-14}$ which should cover the range of variation encountered in practice.

For $\lambda = 10^{-12}$,

$$\sqrt{\alpha} = \sqrt{4\pi\lambda\omega} = 2.10^{-3}$$

and for $\lambda = 10^{-14}$,

$$\alpha = 2.10^{-4}.$$

Correspondingly, $r=2h\sqrt{\alpha}$ has the values 4.0 and 0.4, respectively. Reference to the preceding formulas and curves for J, for $r=4.0$ and $r=0.4$, give

$$J=0.126+i\ 0.168, \qquad \lambda=10^{-12}$$

$$J=0.323+i\ 0.871, \qquad \lambda=10^{-14}$$

whence the corresponding values of Z' are

$$Z'=4\omega.\ (0.126+i\ 0.168),$$

$$Z'=4\omega.\ (0.323+i\ 0.871).$$

These are the "ground return" impedances per unit length in elm. c.g.s. units; to convert to *ohms per mile* they are to be multiplied by the factor 1.61×10^{-4}. Consequently setting $\omega=\pi.10^5$, we get

$$Z'=6.44\pi(1.3+i\ 1.7), \qquad \lambda=10^{-12}$$

$$Z'=6.44\pi\ (3.2+i\ 8.7), \qquad \lambda=10^{-14}.$$

Comparison of these formulas shows that an hundred-fold increase in the resistivity of the ground increases the resistance component of the ground return impedance by the factor 2.5 and increases its reactance only five-fold. That is to say, the ground return impedance is not sensitive to wide variations in the resistivity of the earth, a fortunate circumstance in view of its wide variability and our lack of precise information regarding it.

Induction from Electric Railway Systems

A particularly important application of the preceding analysis is to the problems connected with the disturbances induced in parallel communication lines by alternating current electric railways. Assuming the frequency as 25 c.p.s., we have corresponding to $\lambda=10^{-12}$ and $\lambda=10^{-14}$,

$$\sqrt{\alpha}=0.45\times10^{-4} \text{ and } 0.45\times10^{-5}.$$

Taking the height of the trolley wire as approximately 30 ft., $h=10^3$ and assuming the parallel telephone as the same height above ground and separated by approximately 120 ft., $x=4.10^3$, and

$$r=\sqrt{\alpha}\sqrt{(2h)^2-x^2}$$

$$=4.47\times10^3\sqrt{\alpha},$$

and corresponding to the values of α taken above

$$r=0.2 \text{ and } 0.02 \text{ in round numbers,}$$

while

$$\Theta=\sin^{-1}\frac{4}{\sqrt{20}}=63^\circ\ 30' \text{ approximately.}$$

For both cases, therefore, we can employ, in calculating $J=P+iQ$, the approximate formulas,

$$P=\frac{\pi}{8}-\frac{1}{3\sqrt{2}}r.\cos\Theta+\frac{r^2}{16}\cos 2\Theta\left(.6728+\log\frac{2}{r}\right)+\frac{r^2}{16}\Theta\sin 2\Theta,$$

$$Q=-0.0386+\frac{1}{2}\log\left(\frac{2}{r}\right)+\frac{1}{3\sqrt{2}}r\cos\Theta.$$

For $\lambda=10^{-12}$ and $r=0.2$, this gives

$$J=0.369+i\ 1.135$$

and

$$Z'_{12}=4\omega(0.369+i\ 1.135).$$

The foregoing assumes that the only return conductor is the ground. If, however, an equal and opposite current flows in the rail we must subtract from the foregoing mutual impedance, the mutual impedance between rail and telephone line; that is, the mutual impedance Z'_2 between the telephone line and a conductor at the surface of the earth. For this case

$$\rho''=\sqrt{h^2+x^2}=4.12\times 10^3$$

$$\Theta=\sin^{-1}\frac{4}{\sqrt{17}}=76^\circ$$

$$\cos\Theta=0.242,\ r=0.184 \text{ for } \lambda=10^{-12}.$$

The corresponding value of J is

$$J=0.378+i\ 1.165$$

and the *resultant* mutual impedance between railway and parallel telephone line is,

$$Z'_{12}=4\omega(0.369-0.378+i\ (1.135-1.165)\ )$$
$$=4\omega(0.009-i\ 0.030).$$

The very large reduction in mutual impedance, due to the current in the rail, is striking.

For the case of $\lambda = 10^{-14}$, the corresponding calculations give

$$Z'_{12} = 4\omega(0.391 + i\ 2.27)$$

with ***no current in rail***, and

$$Z'_{12} = 4\omega(-0.001 - i\ 0.002)$$

with *equal and opposite current in rail.* It is evident from these figures that the reduction in mutual impedance, due to the current in the rail, is practically independent of the ground conductivity, at least at the separation specified.